Taylor Models Formalisations: State of the Art

Micaela Mayero

Université Paris 13,
LIPN UMR 7030-LCR team
and
ENS Lyon,
LIP UMR 5668-Arénaire team
http://www-lipn.univ-paris13.fr/~mayero/

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Formal Global Optimisation with Taylor Models

- Roland Zumkeller
- Coq
- 2006
- Refs. : [Zumkeller], [Zumkeller-PhD]

Motivations

- Thomas Hales'proof of Kepler's conjecture
- Formalize global optimisation method based on TM
- Obtain formally proved bound for any multi-variate smooth functions in an efficient way

Global Optimisation

- Finding the minimum and maximum value of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ on a certain domain $[a_1; b_1] \times ... \times [a_n; b_n]$
- From Extremum Criteria to Global Optimisation :
  - Fermat, Euler (1755) : $\nabla\ f\ x = 0$
  - Ramon E. Moore (1962) : Global Optimisation algo. for computers

Concepts

- Interval arithmetic (with Horner evaluation)
- Global optimisation with interval arith.
- Constructives real numbers
- Taylor models
- Composing smooth functions with TM

Implementation of TM in Coq

- Data type :

```
Record
  TaylorModel (degree:nat) (X: list intvl) : Type :=
     TM { approx : Poly R (length X);
          error : intvl
        }.
```

- Computing Taylor coefficients :
  - Composing smooth functions with TM : without invocation of the addition th. chosing “reference point”
  - Symbolic derivative is prohibitively expensive
  - Combinatoric formula are used to obtain the derivatives :

$$inv^{(k)}\ y = (-1)^k \ \frac{k!}{y^{k+1}}$$

$$log^{(k)}\ y = inv^{(k-1)}\ y = (-1)^{k-1} \ \frac{(k-1)!}{y^k}$$

Conclusion of part 1

- To bound the multi-variate polynomials of TM, Horner scheme in interval arithetics are used.
  $\hookrightarrow$ better methods exists (Chebyshev)
- Computation in constructive real numbers suffer from an important performance problem : the cost of evaluating $x + x$ to precision ε is twice that of evaluating x to precision $\varepsilon/2$
- Composition (without addition)
- multi-variate
- PhD thesis of R. Zumkeller (with Benjamin Werner)
- cf Nathalie Revol (Thanks)
- Remarks :
  - No code available
  - cucumber : a global optimisation tool based on Berstein polynomials (in Coq)

A Library of Taylor Models for PVS

- Francisco Cháves, Marc Daumas
- PVS
- 2006
- Refs. : [Chaves-Daumas], [Chaves-PhD]

Motivations

- Solve differential equations
- Library to compute with TM in PVS
- "Library to derive quickly more or less accurate bounds"
- TM to certificate error between f and P

Concepts

- Interval arithmetic
- $\mathbb{Q}$ intervals
- Taylor's theorem with Lagrange remainder adapted to interval arithmetic

Implementation of TM in PVS

- Finite list of monomial functions ("à la Coq")
- Finite sequence of coefficients
- Infinite power series with finite support
- Taylor models :
  $(p, I), \ \{f : X \to \mathbb{R} \setminus \forall x \in X, f(x) - p(x) \in I\}, \ X = [-1, 1]$

```
tm: TYPE = [#P : fs_type, I : Interval #]
```

$+$ axioms and lemmas

Conclusion of part 2

- Significant developpement (PhD thesis of F. Cháves with Marc Daumas and Nathalie Revol)
- Validated by PVS project
- $+,\ -,\ *,\ /,\ \sqrt{},\ exp,\ atan,\ sin,\ sh,\ ch$
- Univariate (futur works : multi-variate)
- Remark : code available

A Taylor Function Calculus for Hybrid System Analysis

- Peter Collins, Milad Niqui, Nathalie Revol
- Coq
- 2010
- Ref. : [Collins-Niqui-Revol]

Motivations

- NB : Ariadne is a tool for the analysis of non linear hybrid systems
- Verification of the numerical algorithms used in Ariadne
- Hybrid systems to model phenomena involving discrete and continuous state space
- Combining Ariadne and Coq : validating Ariadne algorithm using Coq
  (primitives for function *calculus* (kernel) which are based on TM)

Concepts

- Floating-point
- TM based on FP
- [Revol-Makino-Berz]

Implementation of TM in Coq

```
Record Taylor_model :=
  {
   spolynom :> Sparse_polynom;
   error : F
  }.
```

+ basic operations and correctness lemma

NB : `Sparse_polynom` is univariate sparse polynomials with coefficients in F :

```
Record Sparse_polynom :=
  {
   polynom :> list (nat*F);
   polynom_sparse : is_sorted polynom
  }.
```

Conclusion of part 3

- Usable for other tools
- Univariate polynomials
- Floating-point (axiomatized, possible instances)
- Remark : code available

Verifiying the Accurracy of Polynomial Approximations in HOL

- John Harrison
- HOL
- 1997
- Ref. : [Harrison]

Summary

- Polynomial approximations in HOL
- Sceptical approach
- Polynomials in HOL
- Squarefree decomposition
- Sturm's theorem
- Application : P. Tang, 1989
- Several TM in HOL

Conclusion

g : taylor models formalisation

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