Towards Formally Verified Global Optimisation in COQ

Combining CoqApprox and univariate Bernstein polynomials

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Agenda

Goal: formally prove that
$\forall x \in \boldsymbol{I},\ |f(x) - P(x)| \leqslant \epsilon.$

CoqApprox:
$\forall x \in \boldsymbol{I},\ |f(x) - TM(x)| \leqslant \epsilon_1.$

Polynomial Global Optimisation:
$\forall x \in \boldsymbol{I},\ |TM - P(x)| \leqslant \epsilon_2.$

Overview of Bernstein polynomials

$$B_{n,k}(x) := \binom{n}{k}(1-x)^{n-k}x^k, \quad 0 \leqslant k \leqslant n$$

Properties:

$$\forall x \in [0,1],\ B_{n,k}(x) \geqslant 0$$

The $(B_{n,k})_{0 \leqslant k \leqslant n}$ form a basis of $\mathbb{R}[X]_{\leqslant n}$, and a partition of the unity:

$$\forall x \in \mathbb{R},\ \sum_{k=0}^{n} B_{n,k}(x) = 1$$

Sketch of the method

1. Translate the polynomial from $[a, b]$ to $[0, 1]$:

   translate : $p \mapsto q$ such that $\forall x \in [0, 1],\ p(a + x \cdot (b - a)) = q(x)$

2. Convert to the Bernstein basis:

   $$\text{tobern} : q \mapsto r \ \text{ such that } \ \forall k \leqslant n,\ r_k = \sum_{i=0}^{k} q_i \cdot \frac{\binom{k}{i}}{\binom{n}{k}}$$

3. Branch and bound algorithm over $[0, 1]$ (multivariate case: cf. [MN13])

[Demo]

Case study

John Harrison's example [Har97]:

$$\forall x, -\frac{10831}{1000000} \leqslant x \text{ and } x \leqslant \frac{10831}{1000000} \implies \left|(e^x - 1) - \left(x + \frac{8388676}{2^{24}}x^2 + \frac{11184876}{2^{26}}x^3\right)\right| \leqslant \frac{23}{27} \times \frac{1}{2^{33}}.$$

Initial timing: 1h.

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The current prototype in COQ solves it in 3s.

Future work,
Discussion on possible variants or improvements

- Finish the proofs and devise a reflexive tactic
- Handling open intervals and/or unbounded intervals ?
- Choice of parameters ?
- Bernstein with interval coefficients ?
- Multivariate case ?

References

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