Issues, Solutions and Tools in

Formal Proofs for the TMD

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- use annotations for the actual code
  - implement the algorithm in C (for example)
  - write annotation for the behavior of the algorithm
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- use certificates
  - run the algorithm (possibly outside Coq)
  - check in Coq that the result is the root of the function

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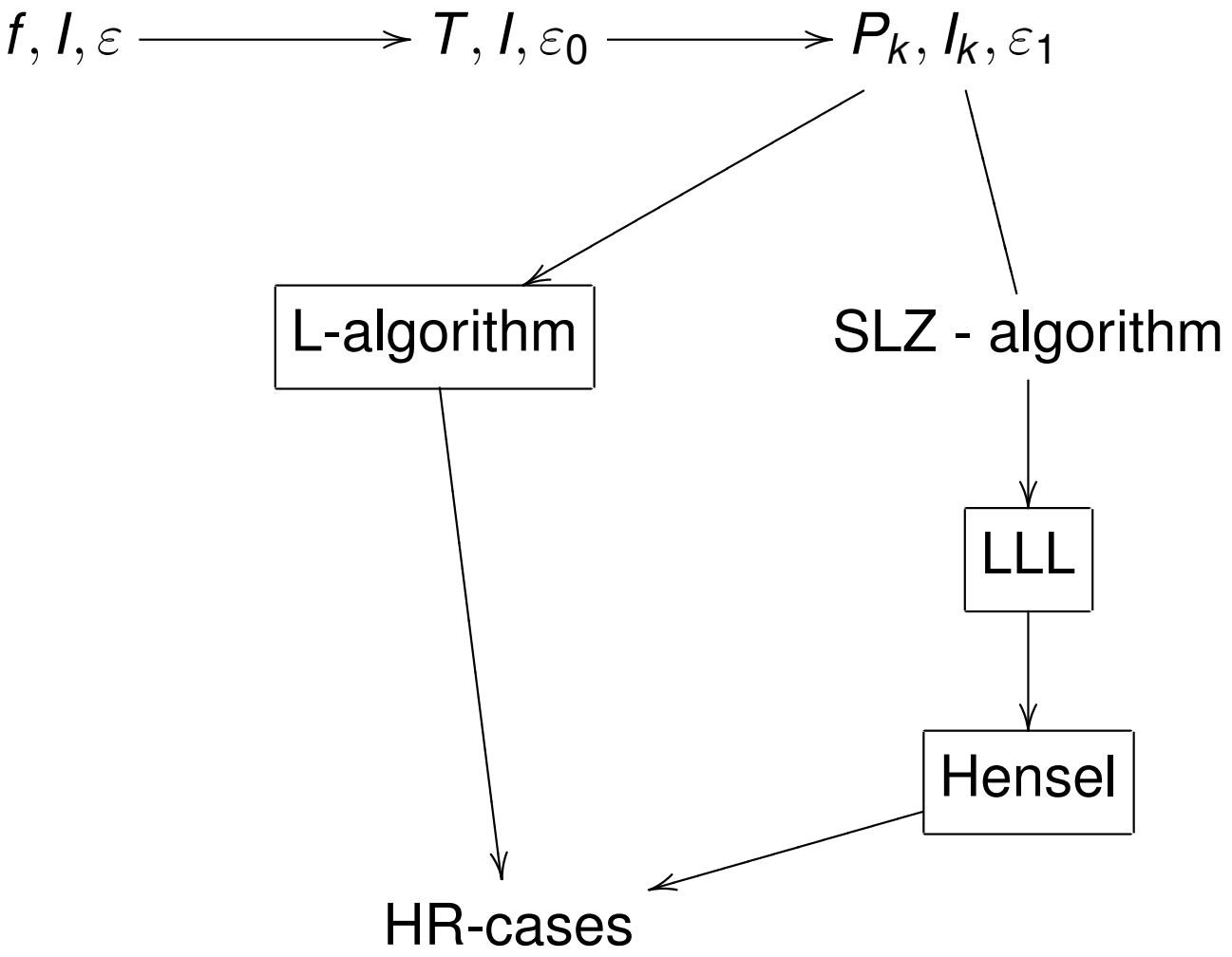
A formal proof view of TMD



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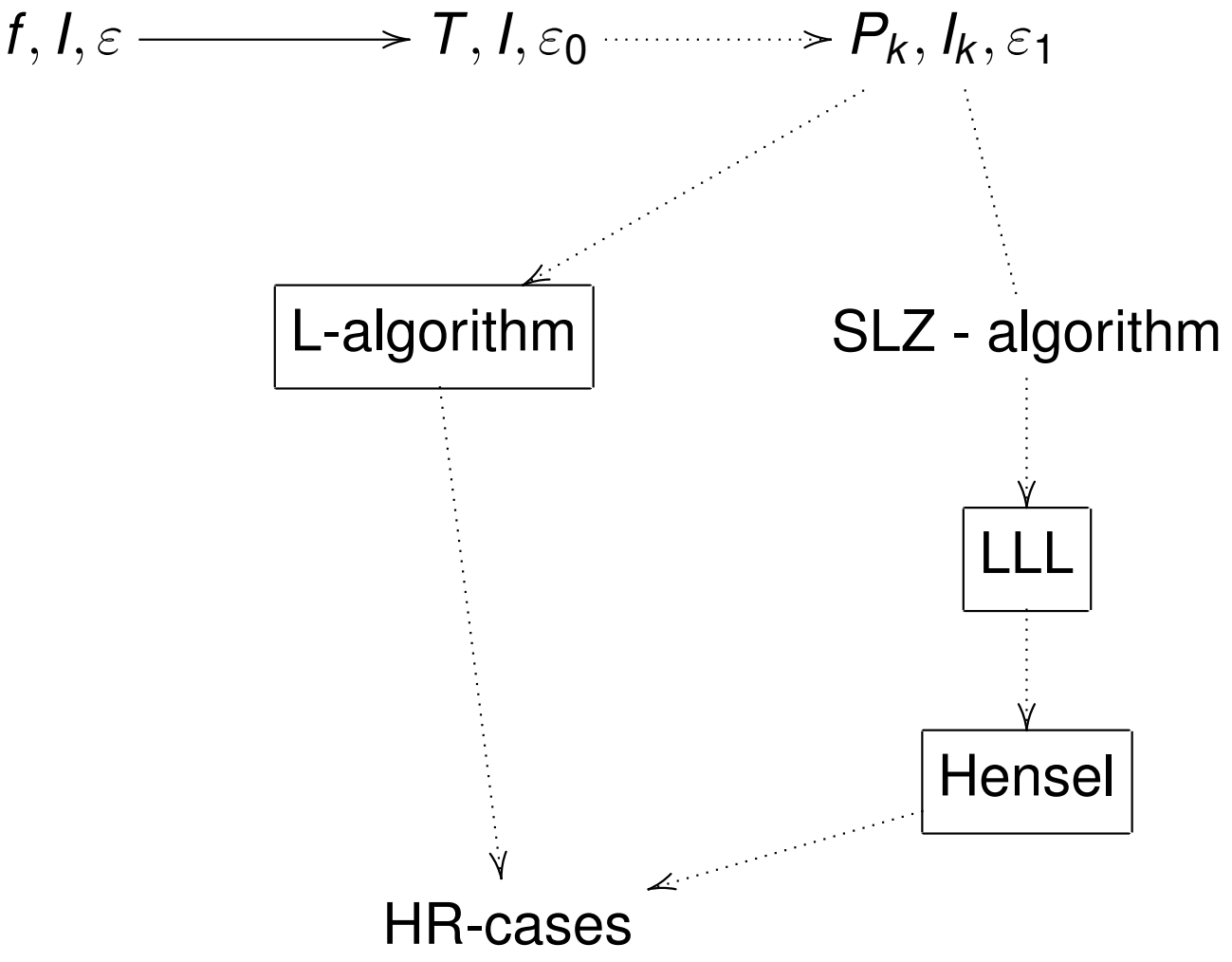

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What solutions?

- formalize Taylor Model algorithm in Coq
- use theoretical properties of polynomial T

Taylor Models (TM)

a TM for f on the interval I is a pair $(T; \Delta)$

- T, polynomial
- Δ, interval
- $\forall x \in I, f(x) - T(x) \in \Delta$

$$T(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$\Delta = \frac{f^{(n+1)}(I)}{n!}(I - x_0)^{n+1}$$

Advantages:

- easy to compute coefficients of T
- easy to compute Δ as the Lagrange reminder
- tight bound for the reminder
- well-studied theoretical properties

Coq formalization of TM

Goal:

- be able to compute T and Δ in Coq
- be able to prove the result is correct, that is

$$\forall x \in I, f(x) - T(x) \in \Delta$$

Libraires for computation in Coq

exact real numbers

- \+ all basic functions
- \+ simpler proofs
- – no evaluation on intervals
- – efficiency ?

FP numbers and intervals

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For exp, on intervals of length 2^{-10}, poly of degree 10:

- exact real numbers 2-3 sec/error computation
- interval FP numbers 1-2 sec/error computation

What choice?

For the library:

- use libraries on FP numbers and intervals

For the Taylor polynomial:

- use interval coefficients ?
- use FP coefficients ?

What about the proofs?

work done for Taylor Models by R. Zumkeller

- remainder as interval with exact real bounds
- no models for basic functions (?)

work for TaMaDi

- prove background results
  - Taylor's theorem
  - correctness of functions in the FP interval library
- formalize and prove correct Taylor models
  - with FP or FP interval coefficients and FP interval reminder

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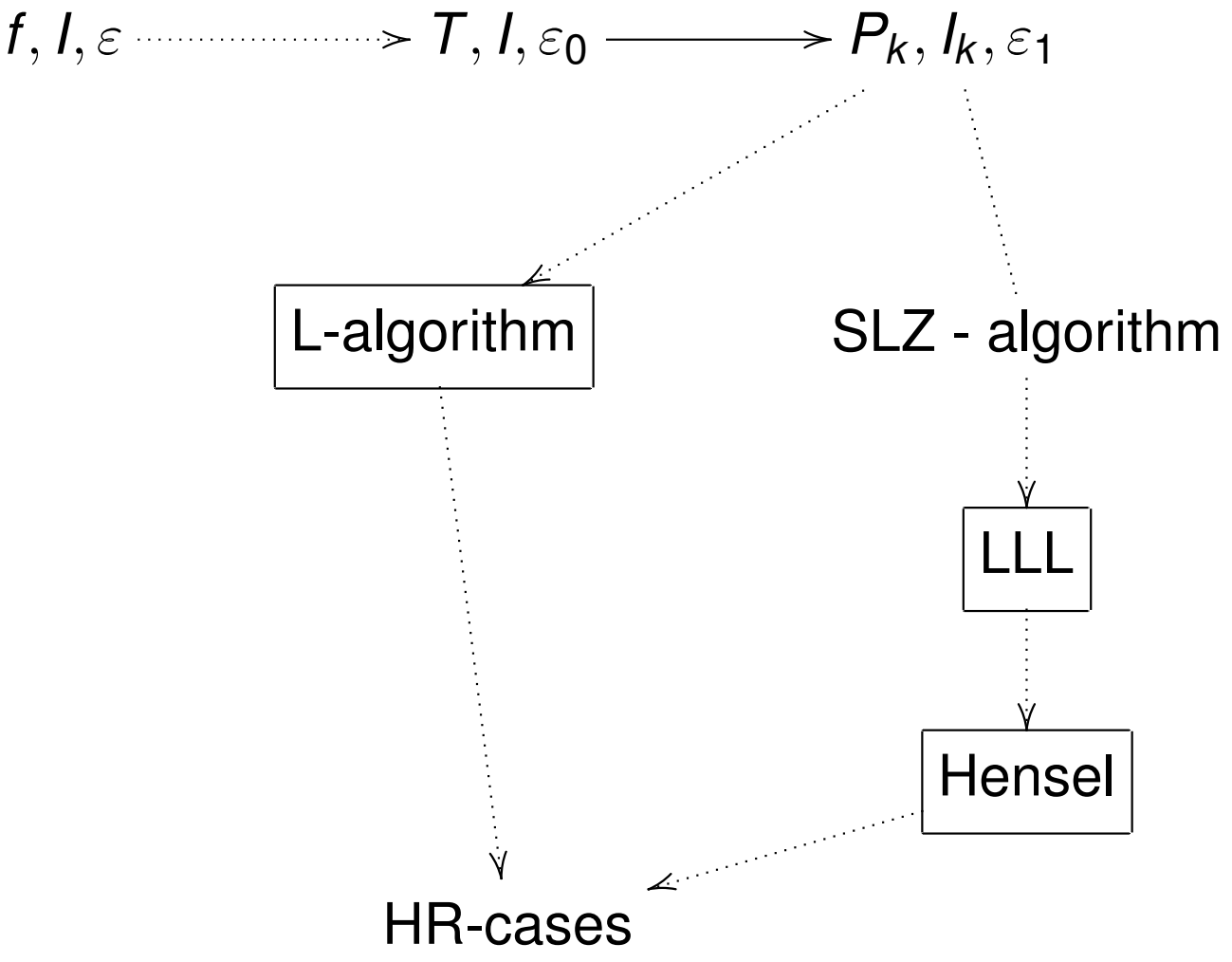

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What solutions?

- sum-of-squares technique (see presentation tomorrow)
- verify the algorithm that computes P_k from T (see presentation today)

Solution 1: sum-of-squares

- certificate based approach

Idea:

$$T^{\{n\}}(X) - P_k^{\{1,2,3\}(X)} = \sum_{i=0}^{n} c_i X^i$$

$$\sum_{i=0}^{n} c_i X^i \leq \varepsilon_1 \Leftrightarrow \varepsilon_1 - \sum_{i=0}^{n} c_i X^i \geq 0 \Leftrightarrow \sum_{i=0}^{n} d_i X^i \geq 0$$

- write this quantity as a sum of squares
- decomposition done by an external tool
- decomposition checked in Coq

Solution 2: verify the algorithm

$$T^{\{n\}}(X) = \sum_{i=0}^{n} a_i X^i$$

- we obtain the coefficients of $P_k^{\{1,2,3\}}$ as $h(a_0, \ldots, a_n)$
- we obtain the error $\| T^{\{n\}} - P_k^{\{1,2,3\}} \| \leq \varepsilon_1(a_0, \ldots, a_n)$

Idea:

- verify these steps in Coq
- evaluate the error directly

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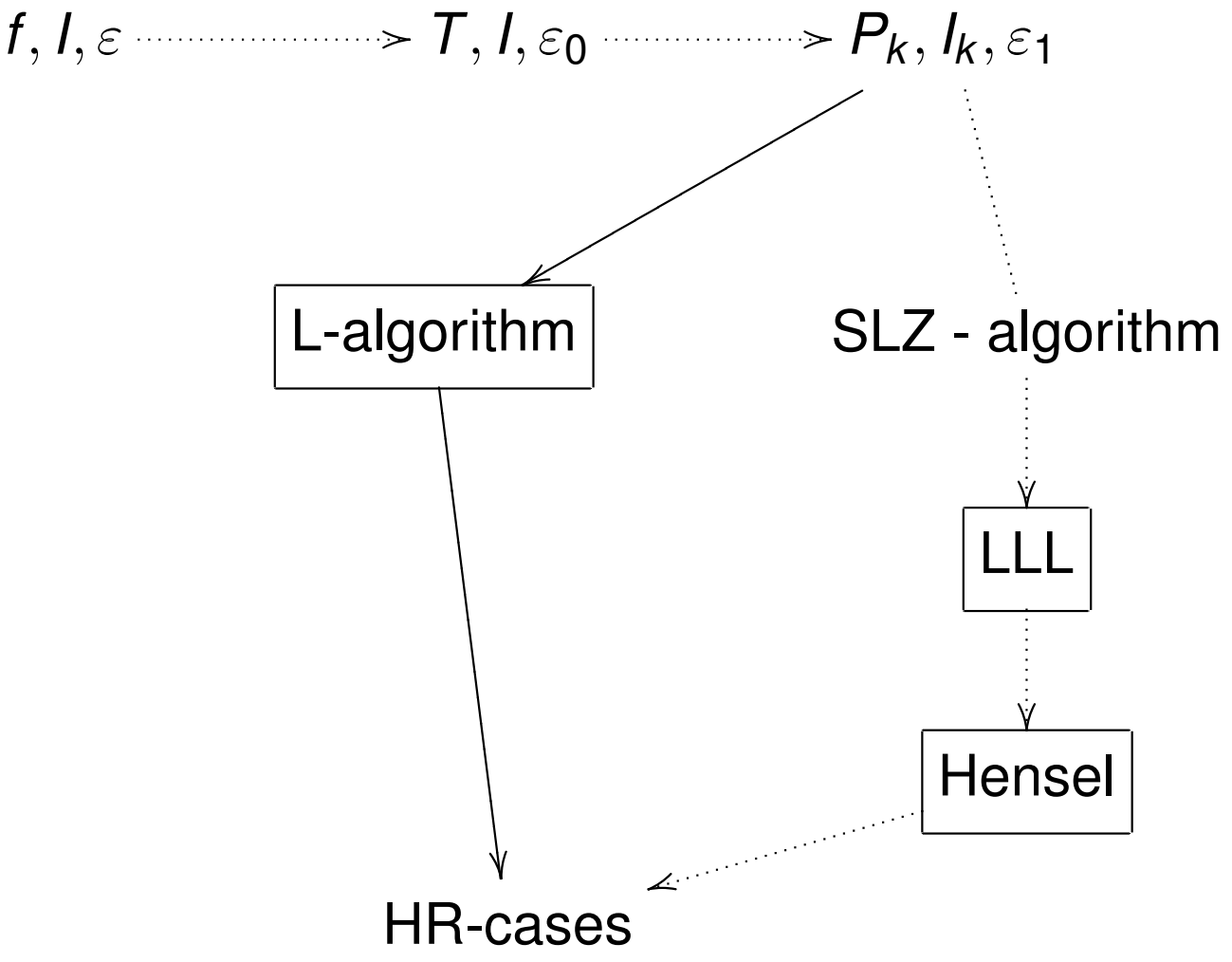
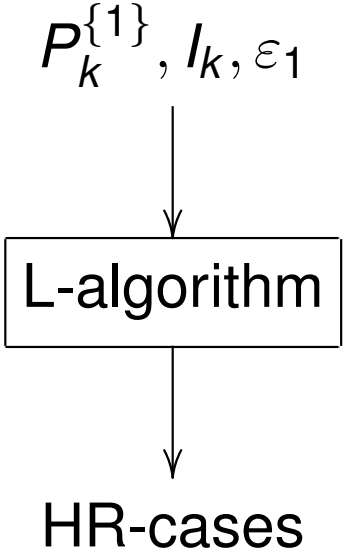


III. The L-algorithm



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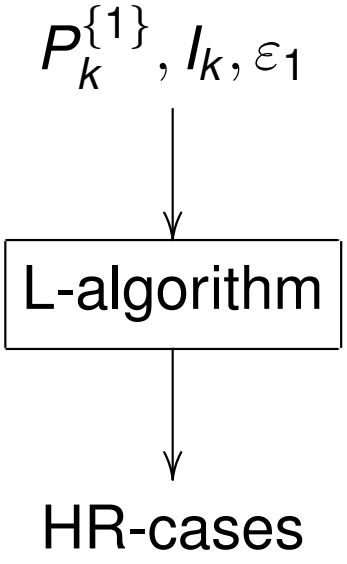

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  - all HR-cases or
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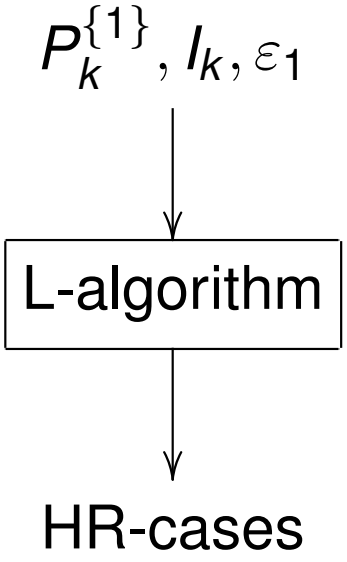

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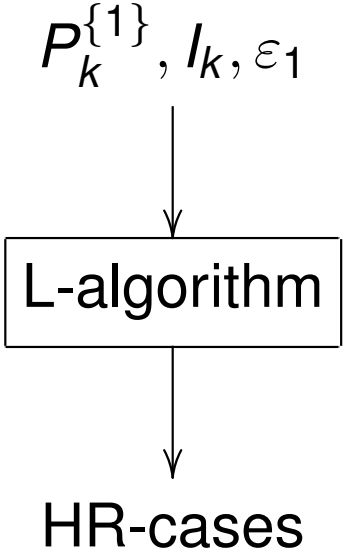

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What solutions?

- prove the annotated C code
- re-think the proof

How do we prove C code?

```
/*@ ensures \result >= x && \result >= y;
    ensures \result == x || \result == y;
*/
int max (int x, int y) { return (x > y) ? x : y; }
```

Frama-C framework

- take annotated C code
- generate proof obligations
- attempt to solve them by automatic and interactive provers

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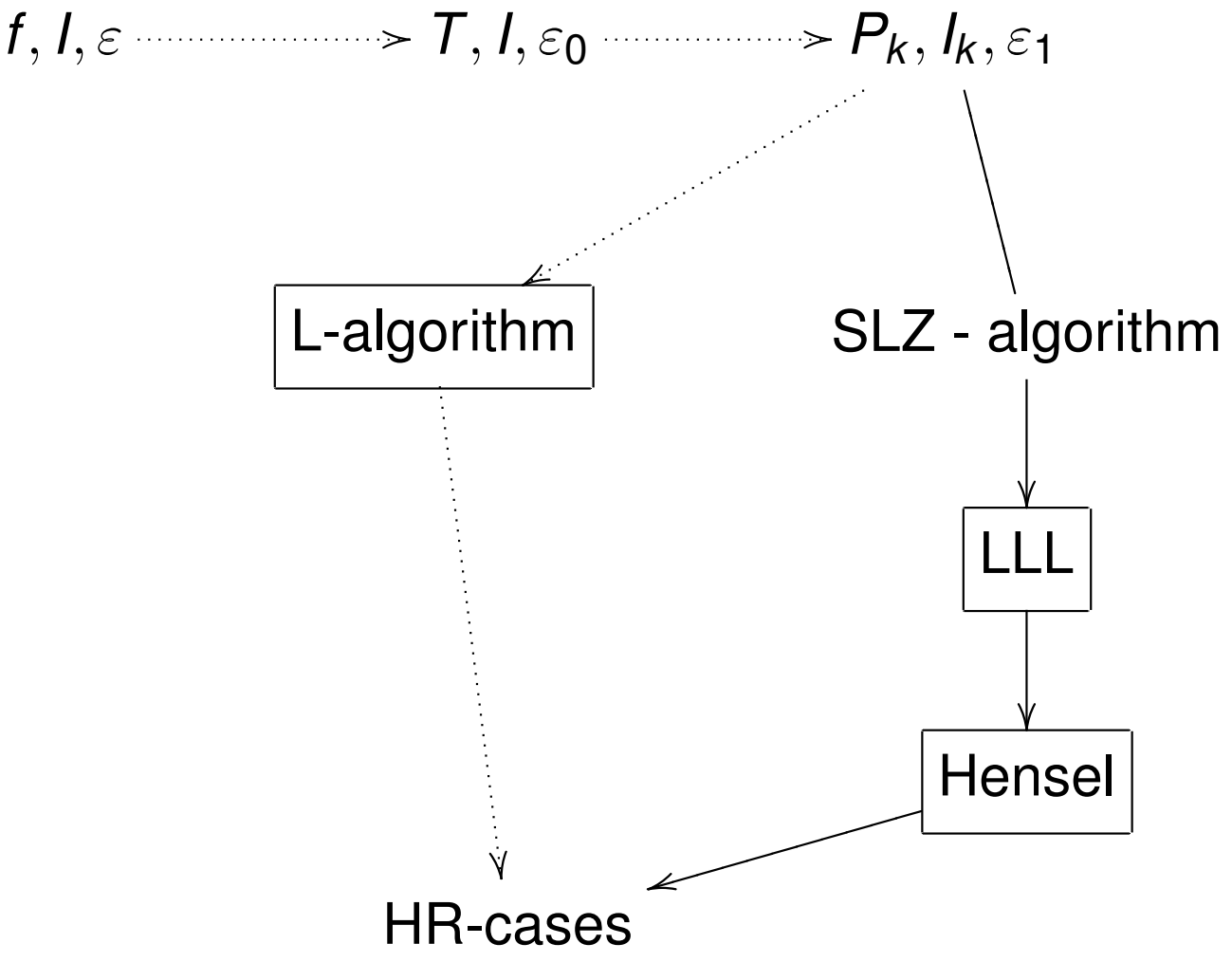

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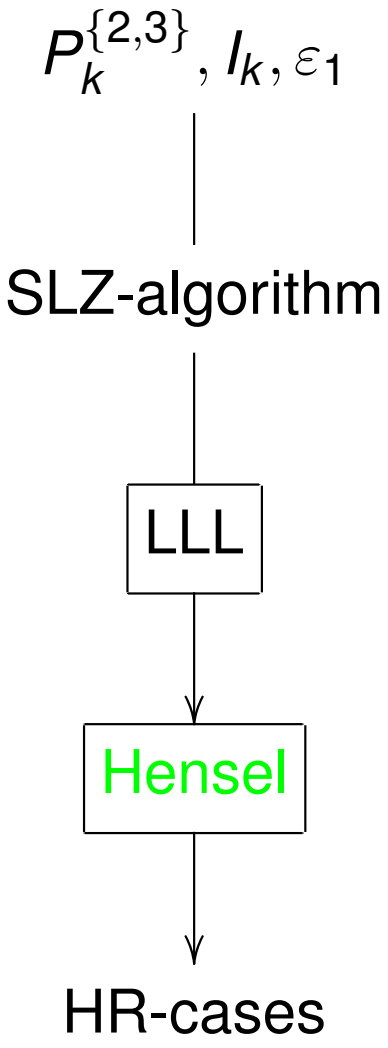

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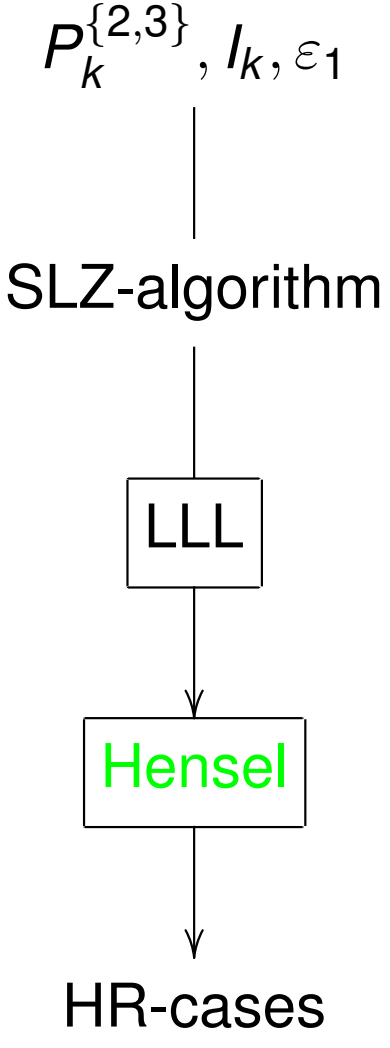

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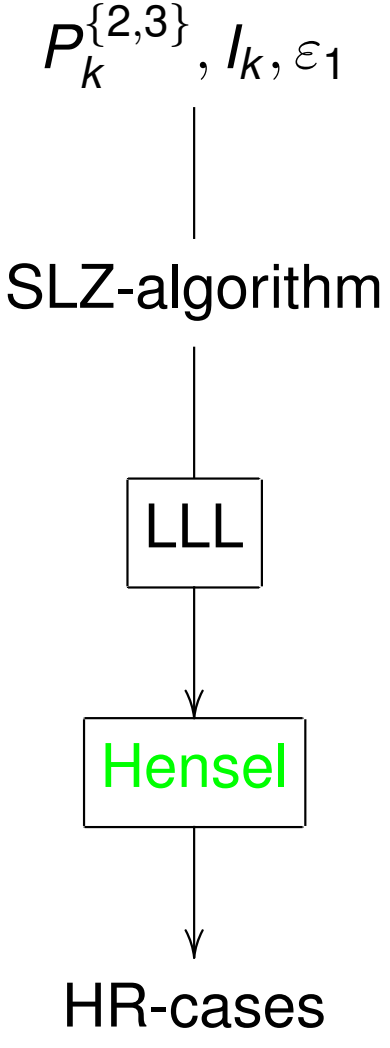

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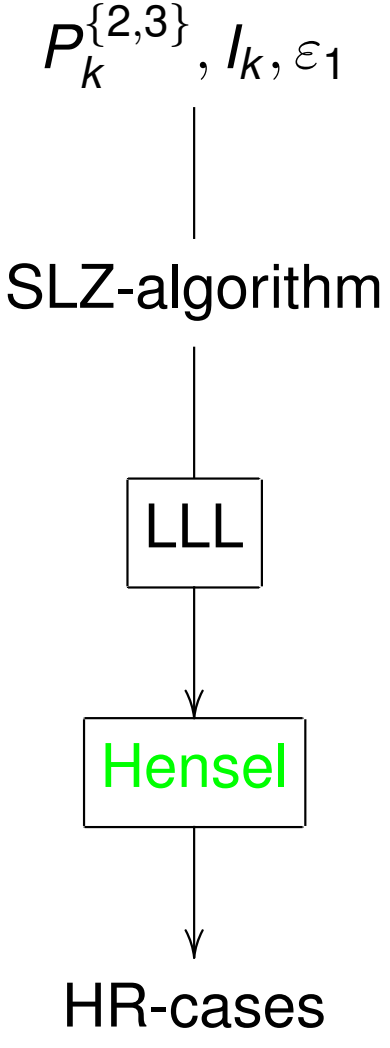

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What solutions?

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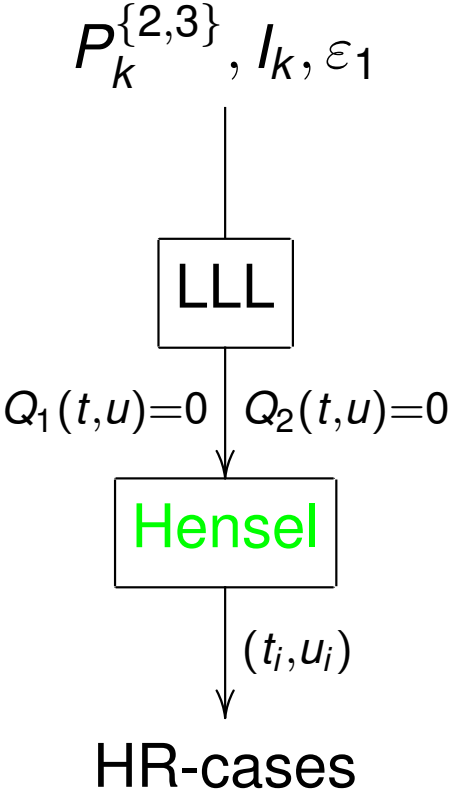
- see paper by Érik (and talk at the first TaMaDi reunion)

A quick reminder on Hensel's lifting

compute all integer roots of a system of 2 bivariate polynomials with integer coefficients



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